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ON THE IRREGULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS.*

BY CLYDE E. LOVE.

1. In the present paper it is proposed to study the integrals of a homogeneous linear differential equation in the neighborhood of an irregular point, particularly with a view to determining the existence and form of their asymptotic developments.† The existence of such developments is well-established provided the roots of the so-called characteristic equation are all distinct,‡ but for the case of multiple roots no general discussion has appeared. In the following work, while the roots are unrestricted as to their order of multiplicity, a certain limitation is placed upon them for the purpose of reducing the algebraic difficulties of the analysis. This limitation is of such a nature that the case of distinct roots is included as a special case of the problem here treated.

The irregular point is taken as the point at infinity, and the independent variable is restricted to real and positive values.

2. As a preliminary step we shall formulate two general theorems§ on linear differential equations, which will form the basis for the subsequent investigation..

Take for consideration the differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0, \tag{1}$$

whose coefficients $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$ are continuous and possess n continuous derivatives for all sufficiently large positive values of x. Let z_1, z_2, $\cdots$, z_n denote n auxiliary functions of x which, together with their first n derivatives, are likewise continuous when x is large and positive, and are such that the determinant

$$Q(x) = \begin{vmatrix} z_1 & z_1' & \cdots & z_1^{(n-1)} \\ z_2 & z_2' & \cdots & z_2^{(n-1)} \\ \vdots & & & \\ z_n & z_n' & \cdots & z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Let $A_r(x)$ be the minor of $Q(x)$ with respect to the element

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* Read before the American Mathematical Society, Dec. 27, 1913.

† In Poincaré's sense: cf. Acta Mathematica, vol. 8 (1886), p. 297.

‡ Cf., for example, Horn, Journal für Mathematik, vol. 138 (1910), pp. 159–191.

§ Cf. Dini, Annali di Matematica, Ser. III, vol. 2 (1899), pp. 297–324.

$z_r^{(n-1)}$. Also define

$$Z_r(x) = z_r a_n - (z_r a_{n-1})' + \cdots + (-1)^{n-1}(z_r a_1)^{(n-1)} + (-1)^n z_r^{(n)}, \qquad r = 1, 2, \cdots, n, \tag{3}$$

and denote by $q(x, x_1)$ the determinant formed from $Q(x)$ by replacing the element $z_r^{(n-1)}(x)$ by $Z_r(x_1)$, $r = 1, \cdots, n$.

Place

$$K(x, x_1) = \frac{(-1)^{n-1}q(x, x_1)}{Q(x)};\tag{4}$$

$$g_r(x) = u_{r,0}(x) = v_{r,0}(x) = \frac{(-1)^{n-1}C_r A_r(x)}{Q(x)}, \quad r = 1, \cdots n,\tag{5}$$

where C_r is an arbitrary constant;

$$u_{r,\lambda}(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} K(x, x_1)K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda)g_r(x_\lambda)dx_\lambda \cdots dx_2 dx_1, \quad \lambda = 1, 2, \cdots,$$

$$v_{r,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty K(x, x_1)K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda)g_r(x_\lambda)dx_\lambda \cdots dx_2 dx_1, \quad \lambda = 1, 2, \cdots.$$

Then we have

THEOREM A: *Suppose that for all values of x greater than some constant, the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} v_{r,\lambda}(x)\tag{6}$$

satisfies the following conditions: (a) the series converges; (b) the series for $y_r(x_1)$ when multiplied by $K(x, x_1)$ may be integrated term by term with respect to x_1 from x to ∞; (c) the series defines a function $y_r(x)$ such that each of the integrals

$$\int_x^\infty y_r(x)Z_s(x)dx, \qquad s = 1, \cdots, n,$$

has a meaning. Then for such values of x the function $y_r(x)$ is an integral of (1).

THEOREM B: *Suppose that a constant a can be found such that for all values of $x > a$ the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} u_{r,\lambda}(x)$$

satisfies the following conditions: (a) the series converges; (b) the series for $y_r(x_1)$ when multiplied by $K(x, x_1)$ may be integrated term by term with respect *to* x_1 *from* a *to* x. *Then for such values of* x *the function* $y_r(x)$ *is an integral of* (1).

To prove Theorem A, place

$$(7)\quad \begin{cases} p_{s,\,0} = z_s, \\ p_{s,\,1} = z_s a_1 - z_s' = z_s a_1 - p_{s,\,0}', \\ \vdots \\ p_{s,\,n-1} = z_s a_{n-1} - p_{s,\,n-2}'; \qquad s = 1, \cdots, n; \end{cases}$$

$$\Phi_s(x) = \int_x^\infty y_r Z_s dx, \qquad s = 1, \cdots, r-1, r+1, \cdots, n;$$

$$\Phi_r(x) = \int_x^\infty y_r Z_r dx + C_r;$$

$$\Delta_r(x) = \begin{vmatrix} p_{1,\,0} & p_{1,\,1} & \cdots & p_{1,\,n-2} & \Phi_1(x) \\ \vdots & & & & \\ p_{n,\,0} & p_{n,\,1} & \cdots & p_{n,\,n-2} & \Phi_n(x) \end{vmatrix},$$

$$\Delta(x) = \begin{vmatrix} p_{1,\,0} & p_{1,\,1} & \cdots & p_{1,\,n-1} \\ \vdots & & & \\ p_{n,\,0} & p_{n,\,1} & \cdots & p_{n,\,n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} Q(x).$$

Now by condition (*b*) equation (6) may be written

$$(8)\qquad y_r(x) = g_r(x) + \int_x^\infty y_r(x_1) K(x, x_1) dx_1.$$

By substituting the values of $g_r(x)$ and $K(x, x_1)$ in (8) we find

$$y_r(x) = \Delta_r(x)/\Delta(x),$$

so that it suffices for our proof to show that this function is an integral of (1).

To do this, consider the system of n functions $\eta_0, \eta_1, \cdots, \eta_{n-1}$ each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$(9)\qquad p_{s,\,0}\eta_{n-1} + p_{s,\,1}\,\eta_{n-2} + \cdots + p_{s,\,n-1}\,\eta_0 = \Phi_s(x), \quad s = 1, \cdots, n.$$

We have at once

$$(10)\qquad \eta_0 = \frac{\Delta_r(x)}{\Delta(x)} = y_r.$$

Upon differentiating equations (9) with respect to x, and making use of (7) and (10), we find

$$(11)\qquad z_s\theta + p_{s,\,0}'\theta_1 + p_{s,\,1}'\theta_2 + \cdots + p_{s,\,n-2}'\theta_{n-1} = 0, \quad s = 1, \cdots, n,$$

where

$$(12)\qquad \theta = \eta_{n-1}' + a_1(x)\eta_{n-2}' + \cdots + a_{n-1}(x)\eta_0' + a_n(x)\eta_0,$$

$$(13)\qquad \begin{cases} \theta_1 = \eta_{n-1} - \eta_{n-2}', \\ \theta_2 = \eta_{n-2} - \eta_{n-3}', \\ \theta_{n-1} = \eta_1 - \eta_0'. \end{cases}$$

The system (11) consists of n homogeneous linear equations in the n quantities $\theta, \theta_1, \cdots, \theta_{n-1}$. By virtue of the relation

$$p_{t,s}' = z_t a_{s+1} - p_{t,s+1}, \qquad t = 1, \cdots, n; \qquad s = 0, 1, \cdots, n-2,$$

the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence by our hypotheses does not vanish for any value of x under consideration. Whence

$$\theta \equiv \theta_1 \equiv \cdots \equiv \theta_{n-1} \equiv 0,$$

or by (10) and (13),

$$\eta_s = y_r^{(s)}, \qquad s = 0, 1, \cdots, n.$$

Substituting in (12), we find

$$y_r^{(n)} + a_1(x)y_r^{(n-1)} + \cdots + a_n(x)y_r \equiv 0,$$

which was to be proved.

The proof of Theorem B follows similar lines, and may be omitted.

These theorems differ only slightly from certain results due to Dini.* He was first led by synthetic processes to the converse theorems, and was then able to establish the direct theorems by reversing his previous reasoning. The appearance of artificiality in the proof as here outlined is chiefly due to the fact that only this reversed line of argument is presented.

3. In the differential equation (1), suppose that the coefficients $a_1(x)$, $a_2(x), \cdots, a_n(x)$ are real or complex functions developable, for large real positive values of x, in asymptotic (or convergent) power series of the form

$$a_r(x) \backsim x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \cdots\right], \qquad r = 1, \cdots, n,$$

k being 0 or a positive integer. Suppose also that the first $n - r$ derivatives of $a_r(x)$ possess asymptotic developments in the same x-region.

We proceed to apply the theorems of § 2 to the problem of finding asymptotic solutions of (1), valid for large real values of x.

Suppose that the characteristic equation

$$\phi(m) = m^n + a_{1,0}m^{n-1} + \cdots + a_{n,0} = 0$$

corresponding to (1) has l different roots $m_1, m_2, \cdots, m_l$, occurring $n_1, n_2, \cdots, n_l$ times respectively: $(n_1 + n_2 + \cdots + n_l = n)$.

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* Loc. cit.

Let us choose n auxiliary functions (cf. § 2) of the form

$$z_{n_1+\cdots+n_{r-1}+q} = x^{-\eta_{r,q}} e^{-f_{r,q}(x)+h_{r,q}(x)}, \qquad r = 1, \cdots, l;\ q = 1, \cdots, n_r, \tag{14}$$

where

$$f_{r,q}(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,q} x^{k+1-\frac{1}{n_r}}}{k+1-\frac{1}{n_r}} + \frac{\beta_{r,q} x^{k+1-\frac{2}{n_r}}}{k+1-\frac{2}{n_r}} + \cdots + \frac{\zeta_{r,q} x^{\frac{1}{n_r}}}{\frac{1}{n_r}}$$

and

$$h_{r,q}(x) = \frac{\theta_{r,q} x^{-\frac{1}{n_r}}}{\frac{1}{n_r}} + \frac{\kappa_{r,q} x^{-\frac{2}{n_r}}}{\frac{2}{n_r}} + \cdots + \frac{\nu_{r,q} x^{-s+1}}{s-1},$$

$\alpha_{r,q}, \beta_{r,q}, \cdots, \nu_{r,q}$ being undetermined constants and s an arbitrary positive integer.

For brevity let us write

$$z_{n_1+\cdots+n_{r-1}+q} = z_{r,q},$$

$$Z_{n_1+\cdots+n_{r-1}+q} = Z_{r,q},$$

$$g_{n_1+\cdots+n_{r-1}+q} = g_{r,q}.$$

Now upon forming $Z_{r,q}(x)$ as given by (3), we find that

$$Z_{r,q}(x) = (-1)^n z_{r,q} x^{nk} [S_{r,q}(x) + x^{-s-k} P_{1,r,q}(x^{1/n_r})], \tag{15}$$

where* $S_{r,q}(x)$ is a polynomial of degree $n_r(s+k)-1$ in x^{-1/n_r}. In $S_{r,q}(x)$, the coefficients of x^{-j/n_r}, $j = 0, 1, \cdots, n_r - 1$, vanish identically because of the fact that

$$\varphi(m_r) = \varphi'(m_r) = \cdots = \varphi^{(n_r-1)}(m_r) = 0.$$

Further, writing this polynomial in the form

$$S_{r,q}(x) = x^{-1}[\mathrm{A}_{r,q} + \mathrm{B}_{r,q} x^{-1/n_r} + \Gamma_{r,q} x^{-2/n_r} + \cdots + \mathrm{N}_{r,q} x^{-s-k+1/n_r}],$$

we have

$$\mathrm{A}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{n_r!} \alpha_{r,q}{}^{n_r} + \psi(m_r),$$

where

$$\psi(m) = a_{1,1} m^{n-1} + a_{2,1} m^{n-2} + \cdots + a_{n,1};$$

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* Throughout the work, the symbol $P(x)$ will denote a function expressible, for large values of x, in the form

$$P(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_p + \epsilon_p(x)}{x^p}, \quad \lim_{x=\infty} \epsilon_p(x) = 0,$$

where p is an arbitrary positive integer.

and

$$\mathrm{B}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\beta_{r,q} + \sigma_{r,q}(m_r, \alpha_{r,q}),$$

$$\Gamma_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\gamma_{r,q} + \tau_{r,q}(m_r, \alpha_{r,q}, \beta_{r,q}),$$

$$\cdot$$

$$\mathrm{N}_{r,q} = \frac{\varphi^{(n_r)}(m_r)}{(n_r - 1)!}\alpha_{r,q}{}^{n_r-1}\nu_{r,q} + \omega_{r,q}(m_r, \alpha_{r,q}, \cdots, \mu_{r,q}),$$

where $\sigma_{r,q}, \tau_{r,q}, \cdots, \omega_{r,q}$ are certain polynomials in the indicated arguments, which it would be useless to write out in full.

Let us now try to determine the coefficients $\alpha_{r,q}, \beta_{r,q}, \cdots, \nu_{r,q}$ by placing*

$$\mathrm{A}_{r,q} = \mathrm{B}_{r,q} = \cdots = \mathrm{N}_{r,q} = 0. \tag{16}$$

For simplicity we shall assume that *no multiple root of the characteristic equation is also a root of the equation* $\psi(m) = 0$, or in other words that there is no value of m satisfying simultaneously the three equations†

$$\varphi(m) = \varphi'(m) = \psi(m) = 0.$$

Under this hypothesis equations (16) can always be solved, and they serve to determine n definite, distinct functions $z_{r,q}$ of the form (14).

4. The quantities $R[f_{r,q}(x)]$, where $R[x]$ denotes the real part of x, will play an important part in what follows. We consider first the case in which these real parts are all distinct, and shall suppose for definiteness that when x is sufficiently large

$$\begin{aligned} R[f_{1,1}(x)] < R[f_{1,2}(x)] < \cdots < R[f_{1,n_1}(x)] \\ < R[f_{2,1}(x)] < \cdots < R[f_{l,n_l}(x)]. \end{aligned} \tag{17}$$

The argument to be used will apply with changes only in notation if these real parts occur in any other order when arranged with respect to their magnitude, so that the assumption (17) involves no loss of generality.

Let n' denote the least common multiple of $n_1, n_2, \cdots, n_l$, and set

$$f(x) = f_{1,1}(x) + f_{1,2}(x) + \cdots + f_{l,n_l}(x),$$

$$\eta = \eta_{1,1} + \eta_{1,2} + \cdots + \eta_{l,n_l} - \frac{(n-1)nk - n + l}{2},$$

$$\Delta = (-1)^{\frac{n(n-1)}{2}}\Pi(m_r - m_s)^{n_r n_s}\Pi(\alpha_{r,q} - \alpha_{r,p}),$$

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* As soon as the constants $\eta_{r,q}$ have been determined, s is to be selected so as to satisfy the inequality (21).

† It is understood, of course, that the case $\psi(m) \equiv 0$ is also excluded, except when the roots of the characteristic equation are all distinct.

where the first product is composed of all the factors $m_r - m_s$ that can be formed with $r = 2, 3, \cdots l$ and $r > s$; the second consists of all factors $\alpha_{r,q} - \alpha_{r,p}$ that can be formed with $r = 1, 2, \cdots, l$; $q = 2, 3, \cdots, n_r$ and $q > p$.

With this notation we find, by (2),

$$Q(x) = \Delta e^{-f(x)} x^{-\eta} P_2(x^{1/n'}),$$

where the constant term in* $P_2(x^{1/n'})$ is unity, so that the hypothesis $Q(x) \neq 0$ for all values of x under consideration is satisfied. Further, with proper choice of the arbitrary constant $C_{r,q}$, we may write by (5),

$$g_{r,q}(x) = e^{f_{r,q}(x)} x^{\rho_{r,q}} P_{3,r,q}(x^{1/n'}), \tag{18}$$

where we have set

$$\rho_{r,q} = \eta_{r,q} + 1 - (n-1)k,$$

and where the constant term in $P_{3,r,q}(x^{1/n'})$ is unity. And by (4) we have

$$K(x, x_1) = \sum_{r=1}^{l} \sum_{q=1}^{n_r} e^{f_{r,q}(x)} x^{\rho_{r,q}} P_{3,r,q}(x^{1/n'}) e^{-f_{r,q}(x_1)} x_1^{-\rho_{r,q}-s} P_{4,r,q}(x_1^{1/n'}). \tag{19}$$

5. We shall now write out by Theorem A a trial solution of (1) corresponding to that one of the functions $g_{r,q}(x)$ whose exponential factor has the least real part, which by (17) is $g_{1,1}(x)$, and show that the conditions of the theorem are satisfied. We have

$$y_{1,1} = g_{1,1}(x) + \sum_{\lambda=1}^{\infty} v_{1,1,\lambda}(x), \tag{20}$$

where

$$v_{1,1,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty K(x, x_1) K(x_1, x_2) \cdots K(x_{\lambda-1}, x_\lambda) g_{1,1}(x_\lambda) dx_\lambda \cdots dx_2 dx_1.$$

Placing

$$\bar{g}_{r,q}(x) = e^{f_{r,q}(x)} x^{\rho_{r,q}},$$

$$\bar{K}(x, x_1) = \sum_{r=1}^{l} \sum_{q=1}^{n_r} e^{f_{r,q}(x)} x^{\rho_{r,q}} e^{-f_{r,q}(x_1)} x_1^{-\rho_{r,q}-s},$$

$$\bar{v}_{r,q,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty \bar{K}(x, x_1) \bar{K}(x_1, x_2) \cdots \bar{K}(x_{\lambda-1}, x_\lambda) \bar{g}_{1,1}(x_\lambda) dx_\lambda \cdots dx_2 dx_1,$$

let us study the behavior, for large values of x, of the function $\bar{v}_{1,1,\lambda}(x)$.

We first choose an arbitrary positive integer p, and then take s so large that*

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* Cf. footnote, p. 149.

$$s > p + 2 + 2\bar{\rho}, \tag{21}$$

where $\bar{\rho}$ is a positive number at least as great as any of the quantities

$$|R[\rho_{r,\, q}]|.$$

Since, by (17),

$$R[f_{1,\,1}(x_\lambda) - f_{r,\,q}(x_\lambda)] \leqq 0$$

for all values of r and q, each of the functions

$$|e^{-f_{r,\,q}(x_\lambda)}x_\lambda^{-\rho_{r,\,q}-s}e^{f_{1,\,1}(x_\lambda)}x_\lambda^{\rho_{1,\,1}}|$$

decreases monotonically for all values of x_λ under consideration, and we have

$$|\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{1,\,1}(x_\lambda)| < n\,|e^{f_{1,\,1}(x_{\lambda-1})}x_{\lambda-1}^{\rho_{1,\,1}}|x_\lambda^{-p-2}.$$

Similarly,

$$|\bar{K}(x_{\lambda-2}, x_{\lambda-1})\bar{K}(x_{\lambda-1}, x_\lambda)g_{1,\,1}(x_\lambda)| < n^2|e^{f_{1,\,1}(x_{\lambda-2})}x_{\lambda-2}^{\rho_{1,\,1}}|x_{\lambda-1}^{-p-2}x_\lambda^{-p-2}.$$

Proceeding in this way, we find

$$\begin{aligned} |\bar{v}_{1,\,1,}\ (x)| < n^\lambda|e^{f_{1,\,1}(x)}x^{\rho_{1,\,1}}| \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{\lambda-1}}^\infty (x_1x_2\cdots x_\lambda)^{-p-2}\,dx_\lambda \\ \cdots dx_2dx_1 < |e^{f_{1,\,1}(x)}x^{\rho_{1,\,1}}|\frac{n^\lambda}{(p+1)^\lambda}\cdot\frac{1}{x^{\lambda(p+1)}}. \end{aligned} \tag{22}$$

By virtue of the defining property of the functions $P(x)$ (see footnote, p. 149), we may take x so large that

$$|v_{1,\,1,\,\lambda}(x)| < 2|\bar{v}_{1,\,1,\,\lambda}(x)|.$$

Whence, by (22) the series $\sum\limits_{\lambda=1}^{\infty} v_{1,\,1,\,\lambda}(x)$ converges absolutely and uniformly for all sufficiently large values of x, so that condition (*a*) of Theorem A is satisfied. Further, by comparison with the geometric series whose general term is twice the right member of (22), it appears that

$$\begin{aligned} \sum_{\lambda=1}^{\infty}|v_{1,\,1,\,\lambda}(x)| < |e^{f_{1,1}(x)}x^{\rho_{1,1}}|\frac{2n}{p+1}\cdot\frac{1}{x^{p+1}}\cdot\frac{1}{1-\dfrac{n}{(p+1)x^{p+1}}} \\ < |e^{f_{1,1}(x)}x^{\rho_{1,1}}|\frac{4n}{p+1}\cdot\frac{1}{x^{p+1}}. \end{aligned} \tag{23}$$

That condition (*b*) is satisfied follows from the results just established, together with the following theorem:†

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* Cf. footnote, p. 150.

† Bromwich, Infinite Series, p. 453.

If $\Sigma f_n(x)$ converges uniformly in any fixed interval $a \leqq x \leqq b$, where b is arbitrary, and if $\varphi(x)$ is continuous for all finite values of x, then

$$\int_a^\infty \varphi(x)\Sigma f_n(x)dx = \Sigma \int_a^\infty \varphi(x) f_n(x)dx,$$

provided that the integral $\int_a^\infty |\varphi(x)|\Sigma|f_n(x)|dx$ is convergent.

For, if we take $a = x$, $x = x_1$, $\varphi(x_1) = K(x, x_1)$, $f_n(x_1) = v_{1,1,\lambda}(x_1)$, it only remains to show that $\int_x^\infty |K(x, x_1)| \sum_{\lambda=0}^{\infty} |v_{1,1,\lambda}(x_1)|dx_1$ converges. But by (4) and (22) this breaks up into n integrals whose convergence is assured by that of the integrals $\int_x^\infty |e^{f_{1,1}(x_1)-f_{r,q}(x_1)}x_1^{\rho_{1,1}-\rho_{r,q}-s}|dx_1$, and each of these converges by reason of (17) and (21).

These same inequalities are sufficient to show that condition (c) holds, if it be recalled that, in (15), we have made

$$S_{r,q}(x) \equiv 0.$$

Hence $y_{1,1}$ as given by (20) is an actual solution of (1).

6. When we attempt to write out a particular integral of (1) corresponding to any other of the functions $g_{r,q}(x)$, the above process fails for the reason that $R[f_{r,q}(x) - f_{1,1}(x)] > 0$ and some of the integrals in $v_{r,q,\lambda}(x)$ diverge. To avoid this difficulty we shall first modify slightly our system of auxiliary functions and then use Theorem B.

The adjoint equation corresponding to (1), viz.

$$z^{(n)} - (a_1 z)^{(n-1)} + \cdots + (-1)^n z a_n = 0, \tag{24}$$

satisfies all the conditions that have been imposed upon (1); it has therefore a particular integral given by (20). The functions $f_{r,q}(x)$ for (24) are obviously the same as for (1), except that all signs are changed. The one having the least real part is therefore $-f_{l,n_l}(x)$, and the corresponding solution of (24) is found by direct computation to be

$$\bar{z}_{l,n_l}x = e^{-f_{l,n_l}(x)}x^{-\eta_{l,n_l}}P_{5,l,n}(x^{1/n}).$$

In our system of auxiliary functions let us replace z_{l,n_l} by this function $\bar{z}_{l,n_l}$, thus making $Z_{l,n_l} \equiv 0$. This leaves $Q(x)$, $g_{r,q}(x)$ and $\bar{g}_{r,q}(x)$ unchanged in essentials, while in $K(x, x_1)$ and $\bar{K}(x, x_1)$ the term for which $r = l$ and $q = n_l$ no longer appears. Now write by Theorem B a trial integral of (1) of the form

$$y_{l,n_l} = g_{l,n_l}(x) + \sum_{\lambda=1}^{\infty} u_{l,n_l,\lambda}(x). \tag{25}$$

Put

$$\bar{u}_{r,q,\lambda}(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} \bar{K}(x, x_1)\bar{K}(x_1, x_2)\cdots\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{r,q}(x_\lambda)dx_\lambda \cdots dx_2 dx_1.$$

Now we have

$$R[f_{l,n_l}(x_\lambda) - f_{r,q}(x_\lambda)] > 0$$

for all values of r and q occurring in $\bar{K}(x_{\lambda-1}, x_\lambda)$. Thus each of the functions $|e^{f_{l,n_l}(x_\lambda)-f_{r,q}(x_\lambda)}x_\lambda{}^{\rho_{l,n_l}-\rho_{r,q}-s}|$ increases monotonically throughout the interval $a \leqq x_\lambda \leqq x_{\lambda-1}$, so that

$$|\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{l,n_l}(x_\lambda)| < n\, |e^{f_{l,n_l}(x_{\lambda-1})}x_{\lambda-1}{}^{\rho_{l,n_l}-(p+2)}|.$$

Similarly,

$$|\bar{K}(x_{\lambda-2}, x_{\lambda-1})\bar{K}(x_{\lambda-1}, x_\lambda)\bar{g}_{l,n_l}(x_\lambda)| < n^2|e^{f_{l,n_l}(x_{\lambda-2})}x_{\lambda-1}{}^{\rho_{l,n_l}-2(p+2)}|.$$

And

$$|\bar{u}_{r,q,\lambda}(x)| < n^\lambda\, |e^{f_{l,n_l}(x)}x^{\rho_{l,n_l}}|\, x^{-\lambda(p+2)} \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\lambda-1}} dx_\lambda \cdots dx_2 dx_1$$

$$< n^\lambda\, |e^{f_{l,n_l}(x)}x^{\rho_{l,n_l}}|\, x^{-\lambda(p+1)}.$$

Whence by reasoning similar to that used in the previous case it appears that the conditions of the theorem are satisfied, and y_{l,n_l} is a solution of (1).

7. We may now form by (25) a second solution of (24), which takes the form

$$\bar{z}_{1,1} = e^{-f_{1,1}(x)}x^{-\eta_{1,1}}P_{5,1,1}(x^{1/n_1}).$$

This being introduced into the auxiliary system in place of $z_{1,1}$, the term in $K(x, x_1)$ and $\bar{K}(x, x_1)$ for which $r = 1$ and $q = 1$ disappears, since now $Z_{1,1} \equiv 0$.

Upon writing out by Theorem A the function*

$$y_{1,2} = g_{1,2}(x) + \sum_{\lambda=1}^{\infty} v_{1,2,\lambda}(x),$$

the difficulty which formerly presented itself is seen to be obviated by our more favorable choice of auxiliary functions, since for all values of r and q now occurring

$$R[f_{1,2}(x) - f_{r,q}(x)] \leqq 0.$$

Having proved by the reasoning of § 5 that $y_{1,2}$ is an integral of (1), we need only introduce a solution $\bar{z}_{l,n_l-1}$ of (24) in place of z_{l,n_l-1} to be able to obtain by Theorem B another solution of (1).

Continuing in this way, we get n integrals of (1) having one or the other of the forms

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* Of course if $n_1 = 1$, we should here form $y_{2,1}$.

$$(26)\qquad \begin{cases} y_{r,\,q} = g_{r,\,q}(x) + \sum\limits_{\lambda=1}^{\infty} u_{r,\,q,\,\lambda}(x), \\[2ex] y_{r,\,q} = g_{r,\,q}(x) + \sum\limits_{\lambda=1}^{\infty} v_{r,\,q,\,\lambda}(x). \end{cases}$$

It is easily shown, and is in fact almost self-evident, that these integrals are linearly independent.

8. In case the quantities $R[f_{r,\,q}(x)]$, while all distinct, occur in another order than that of (17), the above argument evidently serves without modification, merely giving us the integrals in a different order.

In case the real parts are not all distinct, we can obtain at once by Theorem A particular integrals corresponding to each of the functions $f_{r,\,q}(x)$ whose real parts are equal and least. A group of solutions of (24) can then be formed, after which Theorem B serves to determine a group of solutions of (1) corresponding to those functions $f_{r,\,q}(x)$ whose real parts are equal and greatest. The rest of the process is obvious.

In particular, if the quantities $R[f_{r,\,q}(x)]$ are all equal, the complete system of integrals can be written down by Theorem A, at once.

9. It is an immediate consequence of (23) and the similar result for the other solutions (26) that

$$\lim_{x=\infty} x^p \left[\frac{y_{r,\,q} - g_{r,\,q}(x)}{g_{r,\,q}(x)} \right] = 0, \quad r = 1, \cdots, l;\ q = 1, \cdots, n_r.$$

Whence

$$y_{r,\,q} \backsim g_{r,\,q}(x).$$

By direct substitution in the differential equation it appears that we may write

$$y_{r,\,q} \backsim e^{f_{r,\,q}(x)} x^{\rho_{r,\,q}} \sum_{i=0}^{n_r-1} \sum_{j=0}^{\infty} \frac{A_{r,\,q,\,i,\,j}}{x^{j+i/n_r}},$$

where the $A_{r,\,q,\,i,\,j}$ are determinate constants and $A_{r,\,q,\,0,\,0} = 1$.

10. In summary, we may state the following result:

Suppose that the differential equation

$$(1)\qquad y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0$$

satisfies the following conditions for all sufficiently large real positive values of x:

(*a*) *The coefficients* $a_1(x), \cdots, a_n(x)$ *and their first* n *derivatives are continuous.*

(*b*) *The same coefficients are developable in asymptotic* (*or convergent*) *series of the form*

$$a_r(x) \backsim x^{rk} \left[a_{r,\,0} + \frac{a_{r,\,1}}{x} + \cdots \right], \qquad r = 1, \cdots, n,$$

where k is 0 or a positive integer, and the first $n - r$ derivatives of $a_r(x)$ also possess asymptotic developments.

(*c*) *The characteristic equation*

$$m^n + a_{1,\,0}m^{n-1} + \cdots + a_{n,\,0} = 0$$

has l roots $m_1, \cdots, m_l$, occurring respectively $n_1, \cdots, n_l$ times: $(n_1 + \cdots + n_l = n)$ and such that no multiple root of the characteristic equation is also a root of the equation

$$a_{1,\,1}m^{n-1} + a_{2,\,1}m^{n-2} + \cdots + a_{n,\,1} = 0.$$

Then equation (1) *possesses a fundamental system of solutions $y_{r,\,q}$ $(r = 1, \cdots, l;\ q = 1, \cdots, n_r)$ developable asymptotically in the form*

$$y_{r,\,q} \sim e^{f_{r,\,q}(x)} \sum_{i=0}^{n_r-1} x^{\rho_{r,\,q}-i/n_r} \sum_{j=0}^{\infty} \frac{A_{r,\,q\;i,\,j}}{x^j},$$

where $f_{r,\,q}(x)$ is a certain polynomial of degree $n_r(k + 1)$ in x^{1/n_r}, the quantities $\rho_{r,\,q}$ and $A_{r,\,q,\,i,\,j}$ are determinate constants, and $A_{r,\,q,\,0,\,0} = 1$.

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